Numerical analysis of a Langevin equation for systems with infinite absorbing states

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One-dimensional systems with an infinite number of absorbing states exhibit a phase transition that is not fully understood yet. Their static critical exponents are universal and belong in the Reggeon field theory (or directed percolation) universality class. However, exponents associated with the spreading of a localized seed appear to be nonuniversal depending on the nature of the initial condition. We investigate this problem by integrating numerically a non-Markovian Langevin equation proposed recently to describe such phase transitions. We find that the static critical exponents are universal, as expected. On the other hand, the Langevin equation reproduces the nonuniversal behavior observed in microscopic models for exponents associated with the spreading of an initially localized seed and satisfies the generalized hyperscaling relation proposed for those systems. [S1063-651X(97)02210-1]

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Nonequilibrium dynamical phase transitions have attracted a lot of attention in the past decade. Transitions into an *absorbing state* constitute an archetype of such a class of phenomena. These transitions separate a nontrivial *active phase* from an *absorbing phase*, that is, a phase in which the system gets trapped, with probability one, into a microscopic, fluctuation-free state, i.e., an absorbing state, from which it cannot escape.

As conjectured by Janssen and Grassberger [1], many different numerical and analytical studies have established clearly by now that all the systems exhibiting a continuous transition into a *unique* absorbing state, without any other extra symmetry or conservation law, belong into the same universality class, namely, that of directed percolation [2,3]. That same conjecture was extended to include multicomponent systems by Grinstein *et al.* [4]. Among other microscopic models in that broad class are the following: directed percolation [2,3], the contact process [5], catalytic reactions on surfaces [6], branching annihilating random walks with odd parity [7], damage spreading [8], and self-organized systems [9]. The Reggeon field theory (RFT) is the minimal continuous theory capturing the key features of this universality class [10,1], which is often referred to as directed percolation (DP) universality class.

There are microscopic models in which the number of absorbing configurations grows exponentially with system size, being infinity in the thermodynamic limit. Some examples are the pair contact process [11,12], the threshold transfer process [13], and models of catalytic surface reactions such as the dimer reaction [12,14,15]. The critical behavior of these systems is by far more complex and many of their properties remain to be understood. Numerical studies confirm that, as far as the static critical behavior is concerned, all these systems belong in the DP universality class. However, the critical exponents that characterize the spreading of an initially localized perturbation (or seed) in an otherwise absorbing configuration are observed to depend on the nature of the absorbing state and are therefore nonuniversal [11,13].

An analytical attempt to understand the physics of systems with an infinite number of absorbing states was presented in a recent paper [16], in which a continuous field theory characterizing the physics of this phase transition was proposed. That field theory, expressed in terms of an stochastic partial differential (Langevin) equation, reads

$$\partial_t n(\vec{x},t) = \nabla^2 n + rn - un^2 + \alpha n \, \exp\left(-w_1 \int_0^t n(\vec{x},s)\,ds \right) + \sqrt{n}\,\eta(\vec{x},t), \tag{1}$$

where r, u, α, and w_1 are constants, $n(\vec{x},t)$ is the density field at position $\vec{x}$ and time t, and $\eta(\vec{x},t)$ is a Gaussian white noise with zero mean and covariance

$$\langle \eta(\vec{x},t)\,\eta(\vec{x}',t')\rangle = D\,\delta(\vec{x}-\vec{x}')\,\delta(t-t') \tag{2}$$

for some noise amplitude D. Expression (1) is identical to the Langevin equation representing the Reggeon field theory except for the extra non-Markovian term, which at every moment keeps track of all the system time evolution, which is absent in the RFT. Note that Eq. (1) has only one absorbing configuration, namely, $n(\vec{x})=0$ for all $\vec{x}$. It is argued in [16] that all the physical differences between systems with a unique or with infinitely many absorbing states are captured in the additional non-Markovian (exponential) term. The dependence of the microscopic models on the nature of the absorbing states is substitute in Eq. (1) by the parameter α. It is justified in [16] (at least at a mean-field level) that when the initial absorbing state is the *natural* one, that is, an absorbing state generated by the running system, then $\alpha=0$ and the RFT is recovered. This prediction agrees with the microscopic measurements that show that RFT critical exponents are recovered only when the natural absorbing state is considered as environment of an initial seed.

From the present Langevin equation (1) it is concluded in [16] that the exponential term is irrelevant in the renormalization-group sense when computing static critical properties and consequently Eq. (1) belongs in the RFT uni-

TABLE I. Numerical values of the critical exponents for different values of α.

α	η	δ	$\eta+\delta$	z	θ
-0.1	0.18 ± 0.01	0.30 ± 0.01	0.48 ± 0.02	1.26 ± 0.02	0.15 ± 0.01
-0.08	0.21 ± 0.01	0.27 ± 0.01	0.48 ± 0.02	1.25 ± 0.02	0.15 ± 0.01
0.0	0.30 ± 0.01	0.17 ± 0.01	0.47 ± 0.02	1.27 ± 0.02	0.17 ± 0.02
0.2	0.37 ± 0.01	0.11 ± 0.01	0.48 ± 0.02	1.28 ± 0.03	0.17 ± 0.02
0.3	0.43 ± 0.01	0.09 ± 0.01	0.52 ± 0.02	1.28 ± 0.03	0.15 ± 0.02

versality class, as observed in the microscopic models. However, using the same type of arguments, nothing can be concluded about the nature of the transition for the spreading of an initially localized seed and in particular whether it is also RFT-like or is nonuniversal. In fact, for localized-seed type of initial conditions, the extra non-Markovian term cannot be argued to be irrelevant, the reason being that when a point is visited for the first time the exponential term gives a correction to the *mass*, i.e., to the linear term, which affects significantly the spreading of the occupied region over the absorbing space. When a large system is considered, there are points that are visited for the first time at every time step and consequently the non-Markovian term always influences the nature of the spreading. Whether this effect is enough to modify the spreading critical exponents is still an open question.

In this paper we investigate numerically whether the present field theory (1) exhibits the nonuniversal behavior described in one-dimensional microscopic models. To achieve that goal we perform a numerical integration of Eq. (1) with localized initial conditions. Now we present the method and the main results.

The numerical integration of a stochastic differential equation with absorbing states is not a trivial issue. The usual approaches used to study stochastic partial differential equations (see, for example, the integration of the Kardar-Parisi-Zhang equation [17] or of the time-dependent Ginzburg-Landau equation [18]) are not appropriate to deal with equations exhibiting an absorbing state. The reason is that the use of those integration methods without further care generates negative, unphysical values for the field variable and afterward the methods become unstable. Therefore, a more careful scheme is required to integrate Eq. (1).

A numerical algorithm for the numerical study of the RFT has been proposed recently by Dickman [19]. This integration scheme, which we employ in this paper, consists essentially of the following ingredients (see [19] for more details).

(i) There are time and space discretizations. Derivatives are substituted by discrete lattice derivatives.

(ii) The discretization of the field variable at every point is done by defining a minimum nonvanishing density at every lattice site $n_{min}(i,t)$ proportional to Δt. This discretization choice prevents, as explained in [19], the field from becoming negative. The key point is that in this way the modulus of the maximum possible jump of $n(i,t)$ is guaranteed to be smaller than $n(i,t)$ in any case. A new discrete field $m(i,t)$ is defined as $m(i,t)=[n(i,t)/n_{min}(i,t)]$, where $[\,]$ denotes the integer part.

(iii) There is a rescaling of the equation, using its invariance under the following set of transformations: $u\to u'=un_{min}$, $w_1\to w_1'=w_1 n_{min}$, $n(i,t)\to m(i,t)=n(i,t)/n_{min}$, $D\to D'=D/n_{min}$.

(iv) Since $m(i,t)$ can experience only ''quantized'' changes and the discretized equation generates ''jumps'' of arbitrary length [smaller than $m(i,t)$], an intermediate field variable $f(i,t)$ that accumulates the values of small changes at every point is introduced. When the total accumulated variation at a given position i exceeds $n_{min}(i,t)$, the values of $m(i,t)$ and $f(i,t)$ are actualized according to $m(i,t)\to m(i,t)+[f(i,t)]$ and $f(i,t)\to f(i,t)-[f(i,t)]$.

In this way the original stochastic partial differential equation can be studied by analyzing the discrete dynamical system

$$f(i,t+\Delta t)-f(i,t)=\Bigg[rm(i,t)-u'm(i,t)^2+\nabla_d{}^2 m(i,t)+\alpha m(i,t)\exp\Bigg(-w_1'\sum_0^t m\Delta t\Bigg)\Bigg]\Delta t+[\Delta t m(i,t)]^{1/2}\eta(i,t)$$

and

$$m(i,t)\to m(i,t)+[f(i,t)],\qquad f(i,t)\to f(i,t)-[f(i,t)], \tag{3}$$

where ∇_d is the discretized Laplacian operator and $\eta(i,t)$ are independent Gaussian noises with zero mean and variance D'. This process approaches the continuous equation in the double limit $\Delta x\to 0$ and $\Delta t\to 0$. Note that $m(i)=0$ for all i is an absorbing state.

Employing the above-described scheme, extensive computer simulations have been performed. We have fixed $\Delta t=0.02$ and $\Delta x=1$ and considered a lattice of variable size. Other parameters are $u=0.02$, $w_1=1$, and $D'=0.5$. At time $t=0$ the system is initialized with a localized seed given by $m(i,0)=10\Theta(i+4)[1-\Theta(i-4)]$, where Θ is the step function. Initially, the lattice size is fixed to be slightly larger than the central occupied region and it is progressively enlarged as the occupied region spreads out in such a way that the active region does not reach the boundaries. For every set of parameter values we average over up to 40 000 independent runs and study the following magnitudes as a function of time: the sum of the density fields over the whole lattice (averaged over all the runs including those that have reached the absorbing states) $N(t)$, the overall surviving probability $P_s(t)$, that is, the probability that the system has not reached the absorbing state at time t, and the mean-square distance of spreading from the origin of the surviving trials: $R^2(t)=\Sigma_i i^2 m(i,t)/N(t)$. Right at the critical point

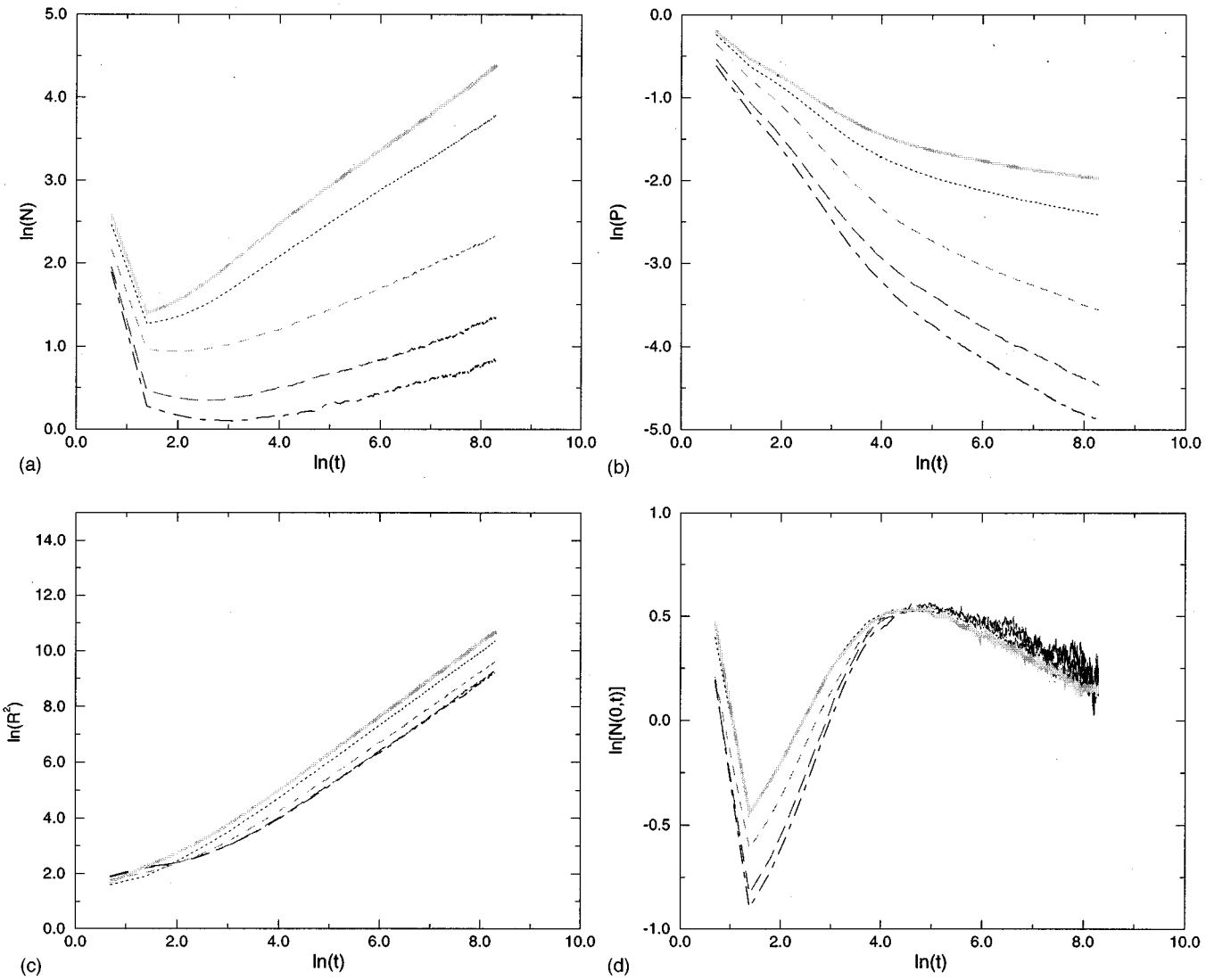

FIG. 1. (a) $\ln[N(t)]$ vs $\ln(t)$, (b) $\ln[P_s(t)]$ vs $\ln(t)$, (c) $\ln[R^2(t)]$ vs $\ln(t)$, and (d) $\ln[N(i=0,t)]$ vs $\ln(t)$ for $r=-0.50583$, $w_1=1$, and (from top to bottom) $\alpha=0.3$ (solid line), $\alpha=0.2$ (dotted line), $\alpha=0.0$ (dashed line), $\alpha=-0.08$ (long-dashed line), and $\alpha=-0.1$ (dot-dashed line), respectively.

there is no characteristic time scale and these magnitudes scale asymptotically as power laws

$$N(t)\propto t^{\eta}, \tag{4}$$

$$P_s(t)\propto t^{-\delta}, \tag{5}$$

and

$$R^2(t)\propto t^{z}. \tag{6}$$

In order to test the numerical procedure we start with the analysis of the RFT, which corresponds to fixing $\alpha=0$. The critical point is located using the idea that the previously introduced magnitudes exhibit a power-law behavior right at the critical point and deviate from it out of criticality.

As at the critical point for long enough times $\eta=d\ \ln[N(t)]/d\ln(t)$, a plot of $d\ \ln[N(t)]/d\ln(t)$ as a function of $1/t$ should give a straight line for small $1/t$ right at the critical point, while for values slightly subcritical or supercritical the curves of that plot should exhibit some curvature. Using that idea, the critical point can be located accurately; it is found to be at $r_c\approx-0.505\,83$, while $r=-0.505\,84$ and $r=-0.505\,82$ are in the absorbing and active phases, respectively. From the extrapolated value of the critical curves at $1/t=0$ we measure the critical exponents $\eta=0.30\pm0.01$, $\delta=0.17\pm0.01$, and $z=1.27\pm0.01$. All of them are consistent within the accuracy limits with the known RFT one-dimensional values $\eta\approx0.312$, $\delta\approx0.159$, and $z\approx1.264$ and with the scaling law for the RFT [2,3], confirming in this way the validity of the integration method.

As was said previously, the dependence of the microscopic models with an infinite number of absorbing states on the nature of the initial absorbing state has it counterpart in the exponential non-Markovian term in Eq. (1) [16]. Therefore, in order to study whether or not that dependence affects the critical behavior in the field theory, we study the Langevin equation for different values of α.

We have performed simulations for $\alpha=0.3$, 0.2, -0.08, and -0.1 and found the following results (see Table I and Fig. 1). (i) We observe no measurable shift in the critical point with respect to the RFT ($\alpha=0$) case: $r_c\approx-0.505\,83$ in all the cases, i.e., the critical point location does not de-

pend on α. (ii) The exponents η and δ are not universal and depend continuously on α. (iii) The exponent z is universal and takes its RFT value. (iv) The combination $\eta+\delta$, which describes the asymptotic behavior of the total number of particles averaged over the surviving runs, is constant for all the α values. The numerical values are presented in Table I.

All the above results are in very good agreement with what is found in the microscopic models. In particular, note that the exponents associated with magnitudes obtained after averaging only over surviving runs (z and $\eta+\delta$) appear to be universal, while those involving the surviving probability (η and δ) present a nonuniversal behavior as α is varied. To further check this conclusion, we have perform other simulations varying the value of w_1. Note that by tuning this parameter the non-Markovian effect can be reduced or amplified, i.e., large values of w_1 imply that the exponential converges rapidly to zero and the memory is small; contrarily, small w_1 values intensify the memory. In all the cases, we observe that the convergence time, that is, the time required to observe the asymptotic regime, depends on w_1, but the critical exponents do not.

Finally, we have also computed the exponent θ that controls the decay of the density at the origin as a function of time (see Table I). As the system evolves the origin gets, on average, deeper and deeper into the cluster of occupied sites. The bulk properties of this cluster have to coincide with those obtained in experiments starting with a homogeneous initial condition. For that type of initial condition and when the system in tuned to the critical point, any initial density decays asymptotically to zero with an exponent θ, i.e., $\langle n(0,t)\rangle \approx t^{-\theta}$. Therefore, measuring θ we can determine whether the bulk critical properties of the non-Markovian theory are RFT-like as predicted in [16].

Results of the simulations are presented in Table I (see the last column). It is clear that within the accuracy limits $\theta \approx 0.16$ is the same for all the values of α, confirming the prediction in [16].

Using the set of measured exponents, we can check the scaling relation proposed for systems with an infinite number of absorbing states [13,20], namely,

$$2(\delta+\eta)+2\theta = dz, \tag{7}$$

for different values of α. In DP it can be shown [13,20] that $\theta=\delta$ and therefore the scaling relation reads $4\delta+2\eta=dz$, but, in general, $\theta\neq\delta$. Note that as Eq. (7) depends on the combination of $\eta+\delta$, z, and θ and these three are constants for different values of α, the hyperscaling relation is satisfied within the accuracy limits in all the cases. Therefore, the one-dimensional Langevin equation with only one absorbing state and a non-Markovian term reproduces remarkably well all the phenomenology of systems with an inifinite number of absorbing states.

We have applied a numerical scheme, introduced by Dickman for the study of stochastic partial differential equations, to the analysis of a non-Markovian Langevin equation. This Langevin equation was recently proposed as a theoretical description of systems with an infinite number of absorbing states. Our numerical study confirms the theoretical prediction that the bulk critical properties are universal and belong in the Reggeon field theory universality class. On the other hand, we have shown that the model reproduces also the, so far not well understood, nonuniversal critical behavior observed in spreading experiments in microscopic models, as well as the hyperscaling relation proposed for that class of system. Therefore, the one-dimensional Langevin equation with only one absorbing state and a non-Markovian term reproduces remarkably well all the phenomenology of systems with an inifinite number of absorbing states. Understanding analytically the nonuniversality from the field theory remains an open challenge. Further analysis and extension of these conclusions to higher dimensions are planned to be presented elsewhere [21].

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